INJECTIVE MODULES AND FP-INJECTIVE MODULES OVER VALUATION RINGS

F. COUCHOT

Abstract. It is shown that each almost maximal valuation ring R, such that every indecomposable injective R-module is countably generated, satisfies the following condition (C): each fp-injective R-module is locally injective. The converse holds if R is a domain. Moreover, it is proved that a valuation ring R that satisfies this condition (C) is almost maximal. The converse holds if $\mathrm{Spec}(R)$ is countable. When this last condition is satisfied it is also proved that every ideal of R is countably generated. New criteria for a valuation ring to be almost maximal are given. They generalize the criterion given by E. Matlis in the domain case. Necessary and sufficient conditions for a valuation ring to be an IF-ring are also given.

In the first part of this paper we study the valuation rings that satisfy the following condition (C): every fp-injective module is locally injective. In his paper [5], Alberto Facchini constructs an example of an almost maximal valuation domain satisfying (C) which is not noetherian and gives a negative answer to the following question asked in [1] by Goro Azumaya: if R is a ring that satisfies (C), is R a left noetherian ring? From [5, Theorem 5] we easily deduce that a valuation domain R satisfies (C) if and only if R is almost maximal and its classical field of fractions is countably generated. In this case every indecomposable injective R-module is countably generated. So, when an almost maximal valuation ring R, with eventually non-zero zerodivisors, verifies this last condition, we prove that R satisfies (C). Conversely, every valuation ring that satisfies (C) is almost maximal.

In the second part of this paper, we prove that every locally injective module is a factor module of a direct sum of indecomposable injective modules modulo a pure submodule. This result allows us to give equivalent conditions for a valuation ring R to be an IF-ring, i.e. a ring for which every injective R-module is flat. It is proved that each proper localization of Q, the classical ring of fractions of R, is an IF-ring.

It is well known that a valuation domain R is almost maximal if and only if the injective dimension of the R-module R is less or equal to one. This result is due to E. Matlis. See [12, Theorem 4]. In the third part, some generalizations of this result are given. Moreover, when the subset Z of zerodivisors of an almost maximal valuation ring R is nilpotent, we show that every uniserial R-module is "standard"(see [7, p.141]).

In the last part of this paper we determine some sufficient and necessary conditions for every indecomposable injective module over a valuation ring R to be countably generated. In particular the following condition is sufficient: $\mathrm{Spec}(R)$ is a countable set. Moreover, when this condition is satisfied, we prove that every ideal of R is countably generated and that every finitely generated R-module is countably cogenerated.

In this paper all rings are associative and commutative with unity and all modules are unital. An R-module E is said to be *locally injective* (or finitely injective, or strongly absolutely pure, [14]) if every homomorphism $A \to E$ extends to a homomorphism $B \to E$ whenever A is a finitely generated submodule of an arbitrary R-module B. As in [6] we say that E is *divisible* if, for every $r \in R$ and $x \in E$, $(0 : r) \subseteq (0 : x)$ implies that $x \in rE$, and that E is *fp-injective*(or absolutely pure) if $\mathrm{Ext}^1_R(F, E) = 0$, for every finitely presented R-module F. A ring R is called *self fp-injective* if it is fp-injective as R-module. An exact sequence $0 \to F \to E \to G \to 0$ is *pure* if it remains exact when tensoring it with any R-module. In this case we say that F is a *pure* submodule of E. Recall that a module E is fp-injective if and only if it is a pure submodule of every overmodule ([17, Proposition 2.6]). A module is said to be *uniserial* if its submodules are linearly ordered by inclusion and a ring R is a *valuation ring* if it is uniserial as R-module. Recall that every finitely presented module over a valuation ring is a finite direct sum of cyclic modules [18, Theorem 1]. Consequently a module E over a valuation ring R is fp-injective if and only if it is divisible. A valuation ring R is *maximal* if every totally ordered family of cosets $(a_i + L_i)_{i\in I}$ has a nonempty intersection and R is *almost maximal* if the above condition holds whenever $\cap_{i\in I} L_i \neq 0$.

We denote $\mathrm{p.d.}_R\, M$ (resp. $\mathrm{i.d.}_R\, M$) the projective (resp. injective) dimension of M, where M is a module over a ring R, $E_R(M)$ the injective hull of M, $\mathrm{Spec}(R)$ the space of prime ideals of R, and for every ideal A of R, $V(A) = \{I \in \mathrm{Spec}(R) \mid A \subseteq I\}$ and $D(A) = \mathrm{Spec}(R) \setminus V(A)$.

When R is a valuation ring, we denote by P its maximal ideal, Z its subset of zerodivisors and Q its classical ring of fractions. Then Z is a prime ideal and $Q = R_Z$.

1. Valuation rings whose fp-injective modules are locally injective

From [5, Theorem 5] we easily deduce the following theorem:

Theorem 1.1. *Let R be a valuation domain. The following assertions are equivalent:*

1. *Every fp-injective module is locally injective.*
2. *R is almost maximal and $\mathrm{p.d.}_R\, Q \leq 1$.*
3. *R is almost maximal and Q is countably generated over R.*
4. *R is almost maximal and every indecomposable injective module is countably generated.*

Proof. (1) $\Rightarrow$ (2). Since Q/R is fp-injective, it is a locally injective module. By [14, Corollary 3.4], Q/R is injective and consequently R is almost maximal by [12, Theorem 4]. From [5, Theorem 5], it follows that $\mathrm{p.d.}_R\, Q \leq 1$.

(2) $\Rightarrow$ (1) is proved in [5, Theorem 5].

(2) $\Leftrightarrow$ (3). See [7, Theorem 2.4, p.76].

(3) $\Leftrightarrow$ (4). By [12, Theorem 4] $E_R(R/A) \simeq Q/A$ for every proper ideal A. Consequently, if Q is countably generated, every indecomposable injective module is also countably generated. □

If R is not a domain then the implication (4) $\Rightarrow$ (1) holds. The following lemma is needed to prove this implication and will be useful in the sequel too.

Lemma 1.2. *Let R be a valuation ring, M an R-module, $r \in R$ and $y \in M$ such that $ry \neq 0$. Then:*

1. $(0 : y) = r(0 : ry)$.
2. *If $(0 : y) \neq 0$ then $(0 : y)$ is finitely generated if and only if $(0 : ry)$ is finitely generated.*

Proof. Clearly $r(0 : ry) \subseteq (0 : y)$. Let $a \in (0 : y)$. Since $ry \neq 0$, $(0 : y) \subset rR$. There exists $t \in R$ such that $a = rt$ and we easily check that $t \in (0 : ry)$. The second assertion is an immediate consequence of the first. □

Theorem 1.3. *Let R be an almost maximal valuation ring. Assume that every indecomposable injective R-module is countably generated. Then every fp-injective R-module is locally injective.*

Proof. Let F be a non-zero fp-injective module. We must prove that F contains an injective hull of each of its finitely generated submodules by [14, Proposition 3.3]. Let M be a finitely generated submodule of F. By [9, Theorem] M is a finite direct sum of cyclic submodules. Consequently, we may assume that M is cyclic, generated by x. Let E be an injective hull of M and $\{x_n \mid n \in \mathbb{N}\}$ a spanning set of E. By [9, Theorem] E is a uniserial module. Hence, for every integer n, there exists $c_n \in R$ such that $x_n = c_n x_{n+1}$. We may suppose that $x = x_0$. By induction on n we prove that there exists a sequence $(y_n)_{n\in\mathbb{N}}$ of elements of F such that $y_0 = x$, $(0 : x_n) = (0 : y_n)$ and $y_n = c_n y_{n+1}$. Since $x_n = c_n x_{n+1}$, $(0 : c_n) \subseteq (0 : x_n) = (0 : y_n)$. Since F is fp-injective, there exists $y_{n+1} \in F$, such that $y_n = c_n y_{n+1}$. We easily deduce from Lemma 1.2 that $(0 : x_{n+1}) = (0 : y_{n+1})$. Now, the submodule of F generated by $\{y_n \mid n \in \mathbb{N}\}$ is isomorphic to E. □

We don't know if the converse of this theorem holds when R is not a domain. However, for every valuation ring R, condition (C) implies that R is almost maximal.

Theorem 1.4. *Let R be a valuation ring. If every fp-injective R-module is locally injective then R is almost maximal.*

Some preliminary results are needed to prove this theorem. The following Lemma will often be used in the sequel. This lemma is similar to [7, Lemma II.2.1].

Lemma 1.5. *Let R be a local commutative ring, P its maximal ideal, U a uniserial R-module, $r \in R$, and $x, y \in U$ such that $rx = ry \neq 0$. Then $Rx = Ry$.*

Proof. We may assume that $x = ty$ for some $t \in R$. It follows that $(1-t)ry = 0$. Since $ry \neq 0$ we deduce that t is a unit. □

Proposition 1.6. *Let U be a uniform fp-injective module over a valuation ring R. Suppose there exists a nonzero element x of U such that $Z = (0 : x)$. Then:*

1. *U is a Q-module.*
2. *For every proper R-submodule A of Q, U/Ax is faithful and fp-injective.*

Proof. (1) For every $0 \neq y \in U$, $(0 : y) = sZ$ or $(0 : y) = (Z : s) = Z$ (see [13]). Hence $(0 : y) \subseteq Z$. If $s \in R \setminus Z$ then the multiplication by s in U is injective. Since U is fp-injective this multiplication is bijective.

(2) If $R \subseteq A$ there exists $s \in R \setminus Z$ such that $sA \subset R$ and there exists $y \in U$ such that $x = sy$. Then $Ax = Asy$ and $(0 : y) = Z$. Consequently we may assume

that $A \subset R$, after eventually replacing A with As and x with y. Let $t \in R$. Since $(0 : t) \subseteq Z$ there exists $z \in U$ such that $x = tz$. Therefore $0 \neq x + Ax = t(z + Ax)$ whence U/Ax is faithful. Let $t \in R$ and $y \in U$ such that $(0 : t) \subseteq (0 : y + Ax)$. Therefore $(0 : t)y \subseteq Ax \subset Qx$. It is easy to check that $(0 : t)$ is an ideal of Q. Since Qx is the nonzero minimal Q-submodule of U we get that $(0 : t) \subseteq (0 : y)$. Since U is fp-injective we conclude that U/Ax is fp-injective too. □

Now, we prove Theorem 1.4.

Proof of Theorem 1.4. If $Z = P$ then R is self fp-injective by [9, Lemma 3]. It follows that R is self injective by [14, Corollary 3.4] and that R is maximal by [11, Theorem 2.3].

Now we assume that $Z \neq P$. In the same way we prove that Q is maximal. From [9, Theorem] it follows that $E_Q(Q/Z) \simeq E_R(R/Z)$ is uniserial over Q and R. Let $H = E_R(R/Z)$ and $x \in H$ such that $Z = (0 : x)$. By Proposition 1.6 H/Px is fp-injective. This module is injective by [14, Corollary 3.4]. Hence $E(R/P) \simeq H/Px$ is uniserial. By [9, Theorem] R is almost maximal. □

From Proposition 1.6 we easily deduce the following corollary which generalizes the second part of [12, Theorem 4].

Corollary 1.7. *Let R be an almost maximal valuation ring, $H = E(R/Z)$ and $x \in H$ such that $Z = (0 : x)$. For every proper and faithful ideal A of R, $H/Ax \simeq E(R/A)$.*

Proof. By [9, Theorem] $E(R/A)$ is uniserial. It follows that its proper submodules are not faithful. We conclude by Proposition 1.6. □

2. Valuation rings that are IF-rings

We begin this section with some results on indecomposable injective modules over a valuation ring. In the sequel, if R is a valuation ring, let $E = E(R)$, $H = E(R/Z)$ and $F = E(R/Rr)$ for every $r \in P$, $r \neq 0$. Recall that, if r and s are nonzero elements of P, then $E(R/Rr) \simeq E(R/Rs)$, (see [13]).

Proposition 2.1. *The following statements hold for a valuation ring R.*

(1) *The modules E and H are flat.*
(2) *The modules E and H are isomorphic if and only if Z is not faithful.*

Proof. First we assume that $Z = P$, whence R is fp-injective. Let $x \in E$, $x \neq 0$, and $r \in R$ such that $rx = 0$. There exists $a \in R$ such that $ax \in R$ and $ax \neq 0$. Then $(0 : a) \subseteq (0 : ax)$, so that there exists $d \in R$ such that $ax = ad$. By Lemma 1.2 $(0 : d) = (0 : x)$, whence there exists $y \in E$ such that $x = dy$. We deduce that $r \otimes x = rd \otimes y = 0$. Hence E is flat. Now if $Z \neq P$, then $E \simeq E_Q(Q)$. Consequently E is flat over Q and R.

Since Q is self-fp-injective, $E_Q(Q/Z) \simeq H$ is flat by [4, Theorem 2.8].

If Z is not faithful there exists $a \in Z$ such that $Z = (0 : a)$. It follows that $H \simeq E(Ra) = E$. □

We state that E and F are generators of the category of locally injective R-modules. More precisely:

Proposition 2.2. *Let R be a valuation ring and G a locally injective module. Then there exists a pure exact sequence: $0 \to K \to I \to G \to 0$, such that I is a direct sum of submodules isomorphic to E or F.*

Proof. There exist a set Λ and an epimorphism $\varphi : L = \oplus_{\lambda\in\Lambda} R_\lambda \to G$, where $R_\lambda = R, \forall\lambda \in \Lambda$. Let $u_\mu : R_\mu \to L$ the canonical monomorphism. For every $\mu \in \Lambda$, $\varphi \circ u_\mu$ can be extended to $\psi_\mu : E_\mu \to G$, where $E_\mu = E, \forall\mu \in \Lambda$. We denote $\psi : \oplus_{\mu\in\Lambda} E_\mu \to G$, the epimorphism defined by the family $(\psi_\mu)_{\mu\in\Lambda}$. We put $\Delta = \hom_R(F, G)$ and $\rho : F^{(\Delta)} \to G$ the morphism defined by the elements of Δ. Thus ψ and ρ induce an epimorphism $\phi : I = E^{(\Lambda)} \oplus F^{(\Delta)} \to G$. Since, for every $r \in P, r \neq 0$, each morphism $g : R/Rr \to G$ can be extended to $F \to G$, we deduce that $K = \ker\phi$ is a pure submodule of I. □

Recall that a ring R is *coherent* if every finitely generated ideal of R is finitely presented. As in [3] we say that R is an *IF-ring* if every injective R-module is flat. From Propositions 2.1 and 2.2 we deduce necessary and sufficient conditions for a valuation ring to be an IF-ring.

Theorem 2.3. *Let R be a valuation ring which is not a field. Then the following assertions are equivalent:*

1. *R is coherent and self-fp-injective*
2. *R is an IF-ring*
3. *F is flat*
4. *$F \simeq E$*
5. *P is not a flat R-module*
6. *There exists $r \in R, r \neq 0$, such that $(0 : r)$ is a nonzero principal ideal.*

Proof.(1)⇒(4). By [3, Corollary 3], for every $r \in P, r \neq 0$, there exists $t \in P, t \neq 0$, such that $(0 : t) = Rr$. Hence $R/Rr \simeq Rt \subseteq R \subseteq E$. We deduce that $F \simeq E$.

(4)⇒(3) follows from Proposition 2.1.

(3)⇒(2) If G is an injective module then by Proposition 2.2 there exists a pure exact sequence $0 \to K \to I \to G \to 0$ where I is a direct sum of submodules isomorphic to E or F. By Propositions 2.1 I is flat whence G is flat too.

(2)⇒(1). See [3, Theorem 2].

(1)⇒(6) is an immediate consequence of [3, Corollary 3].

(6)⇒(5) We denote $(0 : r) = Rt$. If $r \otimes t = 0$ in $Rr \otimes P$ then, by [2, Proposition 13, p. 42], there exist s and d in P such that $t = ds$ and $rd = 0$. Thus $d \in (0 : r)$ and $d \notin Rt$. Whence a contradiction. Consequently P is not flat.

(5)⇒(1). If $Z \neq P$, then $P = \cup_{r\notin Z} Rr$, whence P is flat. Hence $Z = P$. If R is not coherent, there exists $r \in P$ such that $(0 : r)$ is not finitely generated. By Lemma 1.2 $(0 : s)$ is not finitely generated for each $s \in P, s \neq 0$. Consequently, if $st = 0$, there exist $p \in P$ and $a \in (0 : s)$ such that $t = ap$. It follows that $s \otimes t = sa \otimes p = 0$ in $Rs \otimes P$. Whence P is a flat module. We get a contradiction. □

The following theorem allows us to give examples of valuation rings that are IF-rings.

Theorem 2.4. *The following statements hold for a valuation ring R:*

1. *For every $0 \neq r \in P$, R/Rr is an IF-ring.*
2. *For every prime ideal $J \subset Z$, R_J is an IF-ring.*

Proof. (1) For every $a \in P \setminus Rr$ there exists $b \in P \setminus Rr$ such that $r = ab$. We easily deduce that $(Rr : a) = Rb$ whence R/Rr is an IF-ring by Theorem 2.3.

(2) The inclusion $J \subset Z$ implies that there exist $s \in Z \setminus J$ and $0 \neq r \in J$ such that $sr = 0$. If we set $R' = R/Rr$ then $R_J \simeq R'_J$. From the first part and [4, Proposition 1.2] it follows that R_J is an IF-ring. □

The two following lemmas are needed to prove the important Proposition 2.7.

Lemma 2.5. *The following statements hold for a valuation ring R:*

1. *If $Z \neq P$ then $E = PE$.*
2. *If $Z = P$ then $E = R + PE$ and $E/PE \simeq R/P$.*

Proof. (1) If $p \in P \setminus Z$ then $E = pE$.

(2) For every $x \in PE$, $(0 : x) \neq 0$ whence $1 \notin PE$. Let $x \in E \setminus R$. There exists $r \in R$ such that $0 \neq rx \in R$. Since R is self-fp-injective there exists $d \in R$ such that $rd = rx$. By Lemma 1.2 $(0 : d) = (0 : x)$. We deduce that $x = dy$ for some $y \in E$. Then $x \in PE$ if $d \in P$. If d is a unit, in the same way we find $t, c \in R$ and $z \in E$ such that $tc = t(x - d) \neq 0$ and $x - d = cz$. Since $r \in (0 : x - d) = (0 : c)$ then $c \in P$ and $x \in R + PE$. □

Lemma 2.6. *Let R be a valuation ring and U a uniform R-module. If $x, y \in U$, $x \notin Ry$ and $y \notin Rx$, then $Rx \cap Ry$ is not finitely generated.*

Proof. Suppose that $Rx \cap Ry = Rz$. We may assume that there exist $t \in P$ and $d \in R$ such that $z = ty = tdx$. It is easy to check that $(Rx : y - dx) = (Rx : y) = (Rz : y) = Rt \subseteq (0 : y - dx)$. It follows that $Rx \cap R(y - dx) = 0$. This contradicts that U is uniform. □

Proposition 2.7. *Let R be a valuation ring which is not a field. Apply the functor* $\mathrm{Hom}_R(-, E(R/P))$ *to the canonical exact sequence*

$$(S) : 0 \to P \to R \to R/P \to 0. \ Then :$$

1. *If R is not an IF-ring one gets an exact sequence*
$$(S_1) : 0 \to R/P \to E(R/P) \to F \to 0, \ with \ F \simeq \mathrm{Hom}_R(P, E(R/P)),$$
2. *If R is an IF-ring one gets an exact sequence*
$$(S_2) : 0 \to R/P \to E(R/P) \to F \to R/P \to 0, \ with \ PF \simeq \mathrm{Hom}_R(P, E(R/P)).$$

Proof. (1) (S) induces the following exact sequence:

$0 \to R/P \to E(R/P) \to \mathrm{Hom}_R(P, E(R/P)) \to 0$. By Theorem 2.3 P is flat whence $\mathrm{Hom}_R(P, E(R/P))$ is injective. Let f and g be two nonzero elements of $\mathrm{Hom}_R(P, E(R/P))$. There exist x and y in $E(R/P)$ such that $f(p) = px$ and $g(p) = py$ for each $p \in P$. Let Rv be the minimal nonzero submodule of $E(R/P)$. By Lemma 2.6 there exists $z \in (Rx \cap Ry) \setminus Rv$. Then the map h defined by $h(p) = pz$ for each $p \in P$ is nonzero and belongs to $Rf \cap Rg$. Thus $\mathrm{Hom}_R(P, E(R/P))$ is uniform. Now let $a \in R$ such that $af = 0$. It follows that $Pa \subseteq (0 : x) = Pb$ for some $b \in R$. We deduce that $(0 : f) = Rb$. Hence $F \simeq \mathrm{Hom}_R(P, E(R/P))$.

(2) First we suppose that P is not finitely generated. From the first part of the proof it follows that $\mathrm{Hom}_R(P, E(R/P)) \subseteq F$. We use the same notations as in (1). We have $(0 : f) = Rb$ and there exists $c \in P$ such that $(0 : c) = Rb$. Consequently $f \in cF \subseteq PF$. Conversely let $y \in PF$ and $b \in P$ such that $(0 : y) = Rb$. Since $R' = R/Pb$ is not an IF-ring it follows from the first part

that $\mathrm{Hom}_R(P/bP, E(R/P)) \simeq \{x \in F \mid bP \subseteq (0:x)\} \simeq E_{R'}(R'/rR')$ where $0 \neq r \in P/bP$. Hence $\mathrm{Hom}_R(P, E(R/P)) = PF$. We deduce the result from Theorem 2.3 and Lemma 2.5.

If $P = pR$ then $E(R/P) \simeq E \simeq F$. Then multiplication by p induces the exact sequence (S_2). □

Remark 2.8. In [6, Theorem 5.7] A. Facchini considered indecomposable pure-injective modules over a valuation ring R. He proved that $\mathrm{Hom}_R(W, G)$ is indecomposable for every indecomposable injective R-module G, where W is a faithful uniserial module such that $(0:x)$ is a nonzero principal ideal for each $x \in W$. This result implies that $\mathrm{Hom}_R(P, E(R/P))$ is indecomposable when R is an IF-ring and P is faithful.

From Proposition 2.7 we deduce a sufficient and necessary condition for a valuation ring to be an IF-ring. As in [17], the *fp-injective dimension* of an R-module M ($\mathrm{fp-i.d.}_R\, M$) is the smallest integer $n \geq 0$ such that $\mathrm{Ext}_R^{n+1}(N, M) = 0$ for every finitely presented R-module N.

Corollary 2.9. *Let R be a valuation ring. Then the following assertions are equivalent:*

(1) *R is not an IF-ring.*
(2) *$\mathrm{i.d.}_R\, R/P = 1$.*
(3) *$\mathrm{fp-i.d.}_R\, R/Z = 1$.*

Proof. (1) ⇔ (2). It is an immediate consequence of Proposition 2.7.

(1) ⇔ (3). If R is an IF-ring then $Z = P$ and $\mathrm{fp-i.d.}_R\, R/Z > 1$ by Proposition 2.7. Assume that R is not an IF-ring and $Z \neq P$. Let $x \in H$ such that $Z = (0:x)$. By Proposition 1.6 H/Rx is fp-injective. It follows that $\mathrm{fp-i.d.}_R\, R/Z = 1$. □

3. Injective modules and uniserial modules

Proposition 2.7 allows us to give generalizations of well known results in the domain case. This is a first generalization of the first part of [12, Theorem 4].

Theorem 3.1. *Let R be a valuation ring. Then R is almost maximal if and only if F is uniserial.*

Proof. By [9, Theorem] F is uniserial if R is almost maximal. Conversely if F is uniserial, by using the exact sequence (S_1) or (S_2) of Proposition 2.7, it is easy to prove that $E(R/P)$ is uniserial. We conclude by using [9, Theorem]. □

Now we shall prove the existence of uniserial fp-injective modules. This is an immediate consequence of [9, Theorem] when R is an almost maximal valuation ring. The following proposition will be useful for this.

Proposition 3.2. *Let R be a commutative local ring, P its maximal ideal, U a uniserial module with a nonzero minimal submodule S. Then U is a divisible module if and only if it is faithful.*

Proof. First we suppose that U is faithful. Let $0 \neq s \in P$ and $0 \neq y \in U$ such that $(0:s) \subseteq (0:y)$. There exists $t \in R$ such that $x = ty$ where x generates S. Thus $t \notin (0:s)$ and consequently stU is a nonzero submodule of U. It follows that there exists $z \in U$ such that $x = stz$. By Lemma 1.5 $y \in sU$. Conversely

let $0 \neq s \in P$. Then $(0 : s) \subseteq P$ implies that $x \in sU$. We conclude that U is faithful. □

Proposition 3.3. *Let R be a valuation ring such that $Z = P$. Assume that R is coherent, or $(0 : P) \neq 0$, or 0 is a countable intersection of nonzero ideals. Then there exist two uniserial fp-injective modules U and V such that $E(U) \simeq E(R/P)$ and $E(V) \simeq F$. When P is principal then $U \simeq V \simeq R$.*

Proof. If P is principal then R is an IF-ring and it is obvious that $U \simeq V \simeq R$.

Now suppose that R is an IF-ring and P is not finitely generated. Consequently P is faithful. Let $\phi : E(R/P) \to F$ be the homomorphism defined in Proposition 2.7. Thus $F \simeq E$ and $\mathrm{Im}\phi = PF$. It is obvious that $PF \cap R = P$. We put $V = R$ and $U = \phi^{-1}(P)$. Since P is faithful, U is faithful too. It is easy to prove that U is uniserial. Then U is fp-injective by Proposition 3.2.

Now we assume that P is not faithful and not finitely generated. Then R is not an IF-ring. By Corollary 2.9 $\mathrm{i.d.}_R\,(R/P) = 1$. It follows that $R/(0 : P)$ is fp-injective. In this case we put $U = R$ and $V = R/(0 : P)$.

Now we suppose that 0 is a countable intersection of nonzero ideals. We may assume that R is not coherent and P is faithful. By [16, Theorem 5.5] there exists a faithful uniserial R-module U such that $E(U) \simeq E(R/P)$. By Proposition 3.2 U is fp-injective. Let $u \in U$ such that $(0 : u) = P$. Since R is not an IF-ring, then by using Corollary 2.9 it is easy to prove that U/Ru is fp-injective. We put $V = U/Ru$. □

Remark 3.4. By [6, Theorem 5.4] it is obvious that every faithful indecomposable pure-injective R-module is injective if R is a valuation ring such that $(0 : P) \neq 0$. In this case the module W in remark 2.8 doesn't exist. By [6, Theorem 5.7] and Proposition 2.7, PF is the only faithful indecomposable pure-injective R-module which is not injective, when R is an IF-ring and $(0 : P) = 0$.

As in [7, p.15], for every proper ideal A of a valuation ring R we put $A^{\#} = \{s \in R \mid (A : s) \neq A\}$. Then $A^{\#}/A$ is the set of zerodivisors of R/A whence $A^{\#}$ is a prime ideal. In particular $0^{\#} = Z$.

Lemma 3.5. *Let R be a valuation ring, A a proper ideal of R and $t \in R \setminus A$. Then $A^{\#} = (A : t)^{\#}$.*

Proof. Let $a \in (A : t)^{\#}$. If $a \in (A : t)$ then $a \in A^{\#}$. If $a \notin (A : t)$ there exists $c \notin (A : t)$ such that $ac \in (A : t)$. If follows that $act \in A$ and $ct \notin A$ whence $a \in A^{\#}$. Conversely let $a \in A^{\#}$. There exists $c \notin A$ such that $ac \in A$. If $a \in (A : t)$ then $a \in (A : t)^{\#}$. If $a \notin (A : t)$ then $at \notin A$. Since $ac \in A$ it follows that $c = bt$ for some $b \in P$. Since $c \notin A$ it follows that $b \notin (A : t)$. From $abt \in A$ we successively deduce that $ab \in (A : t)$ and $a \in (A : t)^{\#}$. □

If J is a prime ideal contained in Z, we put $ke(J)$ the kernel of the natural map: $R \to R_J$.

Corollary 3.6. *Let R be a valuation ring. Then:*

(1) *For every prime ideal $J \subset Z$ there exist two uniserial fp-injective modules $U_{(J)}$ and $V_{(J)}$ such that $E(U_{(J)}) \simeq E(R/J))$ and $E(V_{(J)}) \simeq E(R_J/rR_J)$, where $r \in J \setminus ke(J)$.*

(2) *If Q is coherent, or Z is not faithful, or 0 is a countable intersection of nonzero ideals, there exist two uniserial fp-injective modules $U_{(Z)}$ and $V_{(Z)}$ such that $E(U_{(Z)}) \simeq H$ and $E(V_{(Z)}) \simeq E(Q/rQ)$, where $0 \neq r \in Z$.*

(3) *If Q is coherent, or Z is not faithful, or 0 is a countable intersection of nonzero ideals, then for every proper ideal A such that $Z \subset A^{\#}$ there exists a faithful uniserial fp-injective module $U_{(A)}$ such that $E(U_{(A)}) \simeq E(R/A)$.*

Proof. (1) is a consequence of Theorem 2.4 and Proposition 3.3.
(2) follows from the above proposition.
(3) is a consequence of (2) and Proposition 1.6. More precisely, since $Z \subset A^{\#}$ we may assume that A is faithful, eventually after replacing A with $(A : a)$ for some $a \in A^{\#}$. Then we put $U_{(A)} = U_{(Z)}/Au$ where $u \in U_{(Z)}$ and $(0 : u) = Z$. □

From Theorem 3.1, Corollary 1.7, Corollary 3.6 and Corollary 2.9 we deduce another generalization of [12, Theorem 4].

Theorem 3.7. *Let R be a valuation ring. Suppose that Q is coherent or maximal, or Z is not faithful, or 0 is a countable intersection of nonzero ideals. Let $U_{(Z)}$ be the fp-injective uniserial module defined in Corollary 3.6 and $u \in U_{(Z)}$ such that $Z = (0 : u)$. Then:*

(1) *If R is not an IF-ring then R is almost maximal if and only if $U_{(Z)}/Ru$ is injective.*

(2) *If R is almost maximal then for every proper and faithful ideal A of R, $U_{(Z)}/Au \simeq E(R/A)$.*

Proof. (1) If $U_{(Z)}/Ru$ is injective then $F \simeq U_{(Z)}/Ru$ is uniserial. By Theorem 3.1 R is almost maximal. Conversely, $U_{(Z)}/Ru$ is a fp-injective submodule of F by Corollary 2.9 and F is uniserial by [9, Theorem]. Let $0 \neq x \in F$. There exists $a \in R$ such that $0 \neq ax \in U_{(Z)}/Ru$. It follows that $\exists y \in U_{(Z)}/Ru$ such that $ax = ay$. By using Lemma 1.5 we deduce that $x \in U_{(Z)}/Ru$. Hence $U_{(Z)}/Ru$ is injective.
(2) is an immediate consequence of Corollary 1.7 □

Let us observe that $U_{(Z)} = Q$ when Z is not faithful and consequently we have a generalization of [12, Theorem 4].
Now this is a generalization of [7, Theorem VII.1.4].

Theorem 3.8. *Let R be an almost maximal valuation ring and suppose that Z is nilpotent. Then:*

(1) *Every indecomposable injective R-module is a faithful factor of Q.*

(2) *An R-module U is uniserial if and only if it is of the form $U \simeq J/I$ where $I \subset J$ are R-submodules of Q.*

(3) *If $I \subset J$ and $I' \subset J'$ are R-submodules of Q then $J/I \simeq J'/I'$ if and only if $I = (I' : q)$ and $J = (J' : q)$ for some $0 \neq q \in Q$.*

Proof. (1) Q is an artinian ring and Z is its unique prime ideal. If A is an ideal such that $A^{\#} = Z$, then A is a principal ideal of Q and $Q \simeq E_R(R/A)$.
(2) We have $E(U) \simeq Q/I$ for some ideal I of R.
(3) We adapt the proof of [7, Theorem VII.1.4]. Suppose that $\phi : J/I \to J'/I'$ is an isomorphism. Since $J/I \simeq sJ/sI$ for every $s \in R \setminus Z$ we may assume that I and I' are proper ideals of R. Then $E(J/I) \simeq E(J'/I')$ implies that $I = (I' : t)$ for

some $0 \neq t \in R$. Since $(J' : t)/(I' : t) \simeq J'/I'$ we may assume that $I' = I$. By [7, Theorem VII.1.4] we may assume that $Z \neq 0$ and consequently that R is maximal by [9, Proposition 1]. The isomorphism ϕ extends to an automorphism φ of $E(J/I)$. Let $a \in Z$ such that $Z = (0 : a)$. If $I^{\#} = Z$ then $Q = E(J/I)$, and if $I^{\#} \neq Z$ then, by eventually replacing I, J and J' with respectively $(I : b), (J : b)$ and $(J' : b)$ for some $b \in R$, we may assume that I is faithful and $Q/Ia = E(J/I)$. By [7, Corollary VII.2.5] φ is induced by multiplication by some $0 \neq q \in Q \setminus Z$. Hence $J = (J' : q)$. In the two cases there exists $x \in E(J/I)$ such that $I = (0 : x) = (0 : \varphi(x))$. By using Lemma 1.2 it follows that $I = (I : q)$. □

From Proposition 2.7 we deduce the following result on the injective dimension of the R-module R.

Proposition 3.9. *Let R be an almost maximal valuation ring such that $R \neq Q$. Then:*

1. *If Q is not coherent then* $\mathrm{i.d.}_R\, R = 2$.
2. *If Q is coherent and not a field then* $\mathrm{i.d.}_R\, R = \infty$. *More precisely, for every R-module M and for every integer $n \geq 1$, then* $\mathrm{Ext}_R^{2n+2}(M, R) \simeq \mathrm{Ext}_R^1(M, Q/Z)$ *and* $\mathrm{Ext}_R^{2n+1}(M, R) \simeq \mathrm{Ext}_R^2(M, Q/Z)$.

Proof. (1) If we apply Proposition 2.7 to Q we deduce that $\mathrm{i.d.}_R\, Q/Z = 1$. Since R is almost maximal, by Corollary 2.9 $\mathrm{i.d.}_R\, R/Z = 1$. By using the exact sequence: $0 \to R/Z \to Q/Z \to Q/R \to 0$, we get $\mathrm{i.d.}_R\, Q/R = 1$. We deduce the result from this following exact sequence: $0 \to R \to Q \to Q/R \to 0$.

(2) By using the same exact sequences as in (1) we easily deduce that $\mathrm{Ext}_R^{p+1}(M, R) \simeq \mathrm{Ext}_R^p(M, Q/Z)$, for each $p \geq 3$. On other hand if we apply Proposition 2.7 to Q we can build an infinite injective resolution of Q/Z with injective terms $(E_n)_{n \in \mathbb{N}}$ such that $E_p \simeq H$ if p is even and $E_p \simeq E(Q/Qa)$, for some $0 \neq a \in R$, if p is odd. Now it is easy to complete the proof. □

Recall that $\mathrm{i.d.}_R\, R = 1$ if and only if R is almost maximal when R is a domain which is not a field ([12, Theorem 4]) and that $\mathrm{i.d.}_R\, R = 0$ if and only if R is almost maximal and $R = Q$ ([11, Theorem 2.3]).

Let U be a uniform module over a valuation ring R. Recall that if x and y are nonzero elements of U such that $(0 : x) \subseteq (0 : y)$ there exists $t \in R$ such that $(0 : y) = ((0 : x) : t)$: see [13]. As in [7, p.144] , we set $U_{\#} = \{s \in R \mid \exists u \in U, u \neq 0 \text{ and } su = 0\}$. Then $U_{\#}$ is a prime ideal and the following lemma holds.

Lemma 3.10. *Let R be a valuation ring and U a uniform R-module. Then for every nonzero element u of U, $U_{\#} = (0 : u)^{\#}$.*

Proof. We set $A = (0 : u)$. Let $s \in A^{\#}$. There exists $t \in (A : s)$ such that $tu \neq 0$. We have $stu = 0$ whence $s \in U_{\#}$. Conversely let $s \in U_{\#}$. There exists $0 \neq x \in U$ such that $s \in (0 : x) \subseteq (0 : x)^{\#} = A^{\#}$. The last equality holds by Lemma 3.5. □

Proposition 3.11. *Let R be a valuation ring and U a uniform fp-injective module. Assume that $U_{\#} = Z = P$. Then:*

1. *U is faithful when P is finitely generated or faithful.*
2. *If P is not faithful and not finitely generated then $ann(U) = (0 : P)$ if $E(U) \not\simeq E(R/P)$ and U is faithful if $E(U) \simeq E(R/P)$.*

Proof. If $E(U) \simeq E(R/P)$ let $u \in U$ such that $(0 : u) = P$. Then for each $0 \neq t \in P$, $(0 : t) \subseteq P$. Hence there exists $z \in U$ such that $tz = u \neq 0$. Hence U is faithful. We assume in the sequel that $E(U) \not\simeq E(R/P)$.

(1) First we suppose that $P = Rp$. Then for every non-finitely generated ideal A it is easy to check that $A = (A : p)$. Consequently, for each $u \in U$, $(0 : u)$ is principal, whence $E(U) \simeq E$. Now we assume that P is faithful. Then P is not principal. For some $0 \neq u \in U$ we put $A = (0 : u)$. Let $0 \neq t \in P$. Then $(0 : t) \subset P$. The equality $A^{\#} = P$ implies that there exists $s \in P \setminus A$ such that $(0 : t) \subset (A : s)$. We have $su \neq 0$ and $(0 : su) = (A : s)$. It follows that there exists $z \in U$ such that $tz = su \neq 0$.

(2) We use the same notations as in (1). If $t \notin (0 : P)$ we prove as in the first part of the proof that there exists $z \in U$ such that $tz \neq 0$. On the other hand for every $s \notin (0 : A)$, $(0 : P) \subseteq sA$. Hence $ann(U) = (0 : P)$. □

From the previous proposition we deduce the following corollary.

Corollary 3.12. *Let R be a valuation ring and U an indecomposable injective R-module. Then the following assertions are true:*

1. *If $Z \subset U_{\#}$ then U is faithful. Moreover, if R is almost maximal then U is a factor module of H.*
2. *If $U_{\#} \subset Z$ then $ann(U) = ke(U_{\#})$.*
3. *If $U_{\#} = Z$ then U is faithful if Z is faithful or finitely generated over Q. If Z is not faithful and not finitely generated over Q then $ann(U) = (0 : Z)$ if $U \not\simeq H$ and H is faithful.*

4. Countably generated indecomposable injective modules

When R is not a domain we don't know if condition (C) implies that every indecomposable injective module is countably generated. However it is possible to give sufficient and necessary conditions for every indecomposable injective R-module to be countably generated, when R is an almost maximal valuation ring.

The following lemmas will be useful in the sequel:

Lemma 4.1. *Let R be a valuation ring and A a proper ideal of R. Then $A \neq \bigcap_{r \notin A} Rr$ if and only if there exists $t \in R$ such that $A = Pt$ and $\bigcap_{r \notin A} Rr = Rt$.*

Proof. Let $t \in (\bigcap_{r \notin A} Rr) \setminus A$. Clearly $Rt = \bigcap_{r \notin A} Rr$, whence $A = Pt$. □

Recall that an R-module M is *finitely* (respectively *countably*) *cogenerated* if M is a submodule of a product of finitely (respectively countably) many injective hulls of simple modules.

Lemma 4.2. *Let R be a valuation ring and A a proper ideal of R. Suppose that R/A is not finitely cogenerated. Consider the following conditions:*

1. *There exists a countable family $(I_n)_{n \in \mathbb{N}}$ of ideals of R such that $A \subset I_{n+1} \subset I_n, \forall n \in \mathbb{N}$ and $A = \bigcap_{n \in \mathbb{N}} I_n$.*
2. *There exists a countable family $(a_n)_{n \in \mathbb{N}}$ of elements of R such that $A \subset Ra_{n+1} \subset Ra_n, \forall n \in \mathbb{N}$ and $A = \bigcap_{n \in \mathbb{N}} Ra_n$.*
3. *R/A is countably cogenerated.*

Then (1) implies (2) and (3) is equivalent to (2).

Proof. If we take $a_n \in I_n \setminus I_{n+1}, \forall n \in \mathbb{N}$, then $A = \bigcap_{n\in\mathbb{N}} Ra_n$. Consequently (1) $\Rightarrow$ (2). It is obvious that $A = \bigcap_{n\in\mathbb{N}} Ra_n$ if and only if $A = \bigcap_{n\in\mathbb{N}} Pa_n$, and this last condition is equivalent to: R/A is a submodule of $\prod_{n\in\mathbb{N}}(R/Pa_n) \subseteq E(R/P)^{\mathbb{N}}$. Hence conditions (2) and (3) are equivalent. □

Lemma 4.3. *Let R be a ring (not necessarily commutative). Then the following conditions are equivalent.*

(1) *Every cyclic left R-module is countably cogenerated.*
(2) *Each finitely generated left R-module is countably cogenerated.*

Proof. Only (1) $\Rightarrow$ (2) requires a proof. Let M be a left R-module generated by $\{x_k \mid 1 \leq k \leq p\}$. We induct on p. Let N be the submodule of M generated by $\{x_k \mid 1 \leq k \leq p-1\}$. The induction hypothesis implies that N is a submodule of G and M/N a submodule of I, where G and I are product of countably many injective hulls of simple left R-modules. The inclusion map $N \to G$ can be extended to a morphism $\phi : M \to G$. Let φ be the composition map $M \to M/N \to I$. We define $\lambda : M \to G \oplus I$ by $\lambda(x) = (\phi(x), \varphi(x))$ for every $x \in M$. It is easy to prove that λ is a monomorphism and conclude the proof. □

Proposition 4.4. *Let R be a valuation ring such that $Z = P$. Consider the following conditions:*

(1) *R and $R/(0 : P)$ are countably cogenerated.*
(2) *P is countably generated.*
(3) *Every indecomposable injective R-module U such that $U_\# = P$ is countably generated.*

Then conditions (1) and (2) are equivalent, and they are equivalent to (3) when R is almost maximal.

Moreover, when the two first conditions are satisfied, every ideal A such that $A^\# = P$ is countably generated and R/A is countably cogenerated.

Proof. (1)$\Rightarrow$(2). We may assume that P is not finitely generated. If $(0 : P) = \cap_{n\in\mathbb{N}} Rs_n$, where $s_n \notin (0 : P)$ and $s_n \notin Rs_{n+1}$ for every $n \in \mathbb{N}$, then, by using [11, Proposition 1.3], it is easy to prove that $P = \cup_{n\in\mathbb{N}}(0 : s_n)$. Since $(0 : s_n) \subset (0 : s_{n+1})$ for each $n \in \mathbb{N}$, we deduce that P is countably generated.

(2)$\Rightarrow$(1). First we assume that P is principal. Then $(0 : P)$ is the nonzero minimal submodule of R, and $(0 : P^2)/(0 : P)$ is the nonzero minimal submodule of $R/(0 : P)$. Hence R and $R/(0 : P)$ are finitely cogenerated. Now assume that $P = \cup_{n\in\mathbb{N}} Rt_n$ where $t_{n+1} \notin Rt_n$ for each $n \in \mathbb{N}$. As above we get that $(0 : P) = \cap_{n\in\mathbb{N}}(0 : t_n)$. Since $(0 : t_{n+1}) \subset (0 : t_n)$ for each $n \in \mathbb{N}$ it follows that $R/(0 : P)$ is countably cogenerated. If $(0 : P) \neq 0$ then R is finitely cogenerated.

(3)$\Rightarrow$(1). It is sufficient to prove that $R/(0 : P)$ is countably cogenerated. We may assume that P is not principal. Then $F \not\simeq E(R/P)$ and $F_\# = P$. Let $\{x_n \mid n \in \mathbb{N}\}$ be a generating subset of F such that $x_{n+1} \notin Rx_n$ for each $n \in \mathbb{N}$. By Proposition 3.11 the following equality holds: $(0 : P) = \cap_{n\in\mathbb{N}}(0 : x_n)$. We claim that $(0 : x_{n+1}) \subset (0 : x_n)$ for each $n \in \mathbb{N}$ else $Rx_{n+1} = Rx_n$. Consequently $R/(0 : P)$ is countably cogenerated.

(1)$\Rightarrow$(3). If P is principal then an ideal A satisfies $A^\# = P$ if and only if A is principal (see the proof of Proposition 3.11). It follows that $U \simeq R$. Now we suppose that P is not finitely generated. Assume that there exists $x \in U$ such that $(0 : x) = (0 : P)$. If $(0 : P) = 0$ then $Rx \simeq R$. It follows that $U = Rx$. If $(0 : P) \neq 0$

then $Rx \simeq R/(0:P)$. Since R is not an IF-ring in this case, $R/(0:P)$ is injective by Corollary 2.9. It follows that $U = Rx$. If $(0:P) \neq 0$ then $E(R/P) \simeq R$. Hence, if U is not finitely generated, we may assume that $(0:P) \subset (0:x)$ for each $x \in U$. We know that $\cap_{n\in\mathbb{N}} Rs_n = (0:P)$ where $s_n \notin (0:P)$ and $s_{n+1} \notin Rs_n$ for each $n \in \mathbb{N}$. Let $(x_n)_{n\in\mathbb{N}}$ a sequence of elements of U obtained by the following way: we pick x_0 a nonzero element of U; by induction on n we pick x_{n+1} such that $(0:x_{n+1}) \subset (0:x_n) \cap Rs_{n+1}$. This is possible since $ann(U) = (0:P)$ by Proposition 3.11. Then we get that $\cap_{n\in\mathbb{N}}(0:x_n) = (0:P)$. If $x \in U$ then there exists $n \in \mathbb{N}$ such that $(0:x_n) \subseteq (0:x)$. Hence $x \in Rx_n$ since U is uniserial.

Now we prove the last assertion. If P is principal then A is also principal and R/A finitely cogenerated. Assume that $P = \cup_{n\in\mathbb{N}} Rs_n$. If $A = Pt$ for some $t \in R$ then A is countably generated and R/A is finitely cogenerated. We may assume that $(A:t) \subset P$ for each $t \in R \setminus A$. Clearly $A \subseteq \cap_{n\in\mathbb{N}}(A:s_n)$. If $b \in \cap_{n\in\mathbb{N}}(A:s_n)$, then $b \in (A:P)$ and it follows that $P \subseteq (A:b)$. Hence $b \in A$, $A = \cap_{n\in\mathbb{N}}(A:s_n)$ and R/A is countably cogenerated. Let $s \in P \setminus (0:A)$. Thus $((0:A):s) = (0:sA) \supset (0:A)$. It follows that $(0:A)^{\#} = P$. Therefore $R/(0:A)$ is countably cogenerated. If $(0:A) = Pt$ for some $t \in R$, then tA is the nonzero minimal ideal of R and by using Lemma 1.5 we show that A is principal. If $(0:A) = \cap_{n\in\mathbb{N}} Rt_n$ then we prove that $A = \cup_{n\in\mathbb{N}}(0:t_n)$, by using [11, Proposition 1.3], when A is not principal. Hence A is countably generated. □

Recall that a valuation ring R is *archimedean* if its maximal ideal P is the only non-zero prime ideal, or equivalently $\forall a, b \in P, a \neq 0, \exists n \in \mathbb{N}$ such that $b^n \in Ra$. By using this last condition we prove that P is countably generated.

Lemma 4.5. *Let R be an archimedean valuation ring. Then its maximal ideal P is countably generated.*

Proof. We may assume that P is not finitely generated. Let $r \in P$. Then there exist s and t in P such that $r = st$ and there exists $q \in P$ such that $q \notin Rs \cup Rt$. Hence for each $r \in P$ there exists $q \in P$ such that $q^2 \notin Rr$. Now we consider the sequence $(a_n)_{n\in\mathbb{N}}$ of elements of P defined in the following way: we choose a nonzero element a_0 of P and by induction on n we choose a_{n+1} such that $a_{n+1}^2 \notin Ra_n$. We deduce that $a_n^{2^n} \notin Ra_0$, for every integer $n \geq 1$. Let $b \in P$. There exists $p \in \mathbb{N}$ such that $b^p \in Ra_0$. Let n be an integer such that $2^n \geq p$. It is easy to check that $b \in Ra_n$. Then $\{a_n \mid n \in \mathbb{N}\}$ generates P. □

By using this lemma, we deduce from Proposition 4.4 the following corollary.

Corollary 4.6. *Let R be a valuation ring and N its nilradical. Consider the following conditions:*

1. *For every prime ideal $J \subseteq Z$, J is countably generated and R/J is countably cogenerated.*
2. *For every prime ideal $J \subseteq Z$ which is the union of the set of primes properly contained in J there is a countable subset whose union is J, and for every prime ideal $J \subseteq Z$ which is the intersection of the set of primes containing properly J there is a countable subset whose intersection is J.*
3. *Every indecomposable injective R-module is countably generated.*

Then conditions (1) and (2) are equivalent and they are equivalent to (3) when R is almost maximal.

Moreover, when the two first conditions are satisfied, every ideal A of Q is countably generated and Q/A is countably cogenerated.

Proof. (3) $\Rightarrow$ (1). For each prime ideal $J \subseteq Z$, R_J is an indecomposable injective R-module. Hence R_J is countably generated. It is obvious that $J = \cap_{n\in\mathbb{N}} Rt_n$, where $t_n \notin J$ for each $n \in \mathbb{N}$ if and only if $\{t_n^{-1} \mid n \in \mathbb{N}\}$ generates R_J. Hence R/J is countably cogenerated. By Proposition 4.4 JR_J is countably generated over R_J. It follows that J is countably generated over R too.

(1) $\Rightarrow$ (3). Since R/J is countably cogenerated and J is countably generated it follows that R_J and JR_J are countably generated. By Proposition 4.4 U is countably generated over R_J and over R too, for every indecomposable injective R-module U such that $U_\# = J$. The result follows from Corollary 1.7.

(1) $\Rightarrow$ (2). Suppose that J is the union of the prime ideals properly contained in J. Let $\{a_n \mid n \in \mathbb{N}\}$ be a spanning set of J such that $a_{n+1} \notin Ra_n$ for each $n \in \mathbb{N}$. We consider $(I_n)_{n\in\mathbb{N}}$ a sequence of prime ideals properly contained in J defined in the following way: we pick I_0 such that $a_0 \in I_0$ and for every $n \in \mathbb{N}$ we pick I_{n+1} such that $Ra_{n+1} \cup I_n \subset I_{n+1}$. Then J is the union of the family $(I_n)_{n\in\mathbb{N}}$. Now if J is the intersection of the prime ideals containing properly J, in a similar way we prove that J is the intersection of a countable family of these prime ideals.

(2) $\Rightarrow$ (1). By Lemma 4.2 we may assume that $V(J)\setminus\{J\}$ has a minimal element I. If $a \in I \setminus J$ then $J = \cap_{n\in\mathbb{N}} Ra^n$. Now we prove that J is countably generated. If $J = N$ then R_J is archimedean. If $J \neq N$, we may assume that $D(J)$ has a maximal element I. Then R_J/IR_J is archimedean too. In the two cases JR_J is countably generated over R_J by Lemma 4.5. On the other hand R/J is countably cogenerated, whence R_J is countably generated over R. Let us observe that $JR_J \simeq J/ke(J)$. It follows that J is countably generated over R too.

Now we prove the last assertion. We put $J = A^\#$. Then $J \subseteq Z$. By Proposition 4.4 A is countably generated over R_J. Since R_J is countably generated over R it follows that A is countably generated over R too. On the other hand, since R_J/AR_J is countably cogenerated, the inclusion $Q/A \subseteq R_J/AR_J$ implies that Q/A is countably cogenerated too, by Lemma 4.2. □

From this corollary we deduce the following results:

Corollary 4.7. *Let R be a valuation ring. Then the following conditions are equivalent:*

1. *Every finitely generated R-module is countably cogenerated and every ideal of R is countably generated.*
2. *For each prime ideal J which is the union of the set of primes properly contained in J there is a countable subset whose union is J, and for each prime ideal J which is the intersection of the set of primes containing properly J there is a countable subset whose intersection is J.*

Proof. It is obvious that (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (1). When R satisfies the condition (D): $Z = P$, this implication holds by Corollary 4.6. Now we return to the general case. Let A be a non-principal ideal of R and $r \in A, r \neq 0$. Then the factor ring R/Rr satisfies the condition (D). Hence A is countably generated and R/A is countably cogenerated. If R is a domain then as in the proof of Corollary 4.6 we show that R is countably cogenerated. If R is not a domain, then Q satisfies (D) and consequently Q is countably cogenerated over Q. By Lemma 4.2 R is countably cogenerated. We conclude by Lemma 4.3. □

Corollary 4.8. *Let R be a valuation ring such that* $\mathrm{Spec}(R)$ *is a countable set. Then:*

(1) *Every finitely generated R-module is countably cogenerated.*
(2) *Every ideal is countably generated.*
(3) *Each fp-injective R-module is locally injective if and only if R is almost maximal.*

Remark 4.9. Let R be a valuation ring and $\Gamma(R)$ its value group. See [15] for the definition of $\Gamma(R)$. If $\mathrm{Spec}(R)$ is countable, then by [15, Theorem 2] and [8, Lemma 12.11, p. 243] we get that $\aleph_0 \leq |\Gamma(R)| \leq 2^{\aleph_0}$. Conversely if $|\Gamma(R)| \leq \aleph_0$ it is obvious that every ideal is countably generated and that each finitely generated R-module is countably cogenerated.

Let us observe that if an almost maximal valuation ring R satisfies the conditions of Corollary 4.6, then every indecomposable injective R-module U such that $U_{\#} \subset Z$ is flat since R_J is an IF-ring. It follows that $\mathrm{p.d.}_R U \leq 1$ by [8, Proposition 9.8 p.233]. On the other hand, when R is a valuation domain that satisfies (C), first it is proved that $\mathrm{p.d.}_R Q = 1$ and afterwards, by using [7, Theorem 2.4 p.76], or by using methods of R.M. Hamsher in [10], it is shown that Q is countably generated. When R is not a domain it is possible to prove the following proposition:

Proposition 4.10. *Let R be a valuation ring and J a nonmaximal prime ideal. The following assertions are equivalent:*

(1) *R_J is countably generated.*
(2) $\mathrm{p.d.}_R R_J = 1$.

Proof. (1)⇒(2). By [8, Proposition 9.8 p.233] , $\mathrm{p.d.}_R R_J = 1$ since R_J is flat and countably generated.

(2)⇒(1). If $R' = R/ke(J)$ then $\mathrm{p.d.}_{R'} R_J = 1$ since R_J is flat. Then, after eventually replacing R with R', we may assume that every element of $R \setminus J$ is not a zerodivisor. We use similar methods as in [10]. If S is a multiplicative subset (called semigroup in [10]) of R then $R \setminus S$ is a prime ideal. First, if $s \in P \setminus J$ we prove there exists a prime ideal J' such that $s \notin J'$, $J \subseteq J'$, $R_{J'}$ is countably generated and $\mathrm{p.d.}_R (R_J/R_{J'}) \leq 1$: we do a similar proof as in [10, Proposition 1.1]. Now we prove that $R_{J'}/tR_{J'}$ is free over R/tR, for every non-zerodivisor t of R: we do as in the proof of [10, Proposition 1.2] . Suppose that $J' \neq J$ and let $t \in J' \setminus J$. Since s divides $R_{J'}/tR_{J'}$ and that this module is free over R/tR, we get that $R_{J'} = tR_{J'}$, whence a contradiction. □

References

[1] G. Azumaya, Finite splitness and finite projectivity. *J. Algebra* **106** (1987), 114–134
[2] N. Bourbaki, Algèbre commutative, Chapitre 1, Hermann, Paris, 1961.
[3] R. Colby, Flat injective modules. *J. Algebra* **35** (1975), 239–252
[4] F. Couchot, Exemples d'anneaux auto-fp-injectifs. *Comm. Algebra* **10(4)** (1982), 339–360.
[5] A. Facchini, Absolutely pure modules and locally injective modules, in: Lect. Notes Pure Appl. Math.,vol. 153, 105–109. *Commutative ring theory: Proceedings of the Fès international conference, April 20-24, 1992. ed: P.J Cahen*, Marcel Dekker, New York.
[6] A. Facchini, Relative Injectivity and Pure-injective Modules over Prüfer Rings. *J. Algebra* **110** (1987), 380–406.
[7] L. Fuchs and L. Salce, Modules over valuation domains, in: Lecture Notes in Pure an Appl. Math., vol. 97, Marcel Dekker), New York, 1985.

[8] L. Fuchs and L. Salce, Modules over Non-Noetherian Domains, Mathematical Surveys and Monographs, vol. 84, American Mathematical Society, Providence, 2001.

[9] D.T. Gill, Almost maximal valuation rings. *J. London Math. Soc.* **4** (1971), 140–146.

[10] R.M. Hamsher, On the structure of a one-dimensional quotient field. *J. Algebra* **19** (1971), 416–425.

[11] G.B. Klatt and L.S. Levy, Pre-self injective rings. *Trans. Amer. Math. Soc.* **137** (1969), 407–419.

[12] E. Matlis, Injective modules over Prüfer rings. *Nagoya Math. J.* **15** (1959), 57–69.

[13] M. Nishi, On the ring of endomorphisms of an indecomposable module over a Prüfer ring. *Hiroshima Math. J.* **2** (1972), 271–283.

[14] V.S Ramamurthi and K.M Rangasmy, On finitely injective modules. *J. Aust. Math. Soc.* **XVI(part 2)**, (1973), 239–248.

[15] T. S. Shores, On generalized valuation rings. *Michigan Math. J.* **21**, (1974), 405–409.

[16] T.S. Shores and W.J. Lewis, Serial modules and endomorphism rings. *Duke Math. J.* **41**, (1974), 889–909.

[17] B. Stenström, Coherent rings and FP-injective modules. *J. London Math. Soc.* **2(2)**,(1970), 323–329.

[18] R.Warfield, Decomposability of finitely presented modules. *Proc. Amer. Math. Soc.* **25**, (1970), 167–172.